

Self-Similarity in Decaying Two-Dimensional Stably Stratified Adjustment

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(Dated: February 2, 2008)

The evolution of large-scale density perturbations is studied in a stably stratified, two-dimensional flow governed by the Boussinesq equations. As is known, initially smooth density (or temperature) profiles develop into fronts in the very early stages of evolution. This results in a frontally dominated k^{-1} potential energy spectrum. The fronts, initially characterized by a relatively simple geometry, spontaneously develop into severely distorted sheets that possess structure at very fine scales, and thus there is a transfer of energy from large to small scales. It is shown here that this process culminates in the establishment of a $k^{-5/3}$ kinetic energy spectrum, although its scaling extends over a shorter range as compared to the k^{-1} scaling of the potential energy spectrum. The establishment of the kinetic energy scaling signals the onset of enstrophy decay which proceeds in a mildly modulated exponential manner and possesses a novel self-similarity. Specifically, the self-similarity is seen in the time invariant nature of the probability density function (PDF) associated with the normalized vorticity field. Given the rapid decay of energy at this stage, the spectral scaling is transient and fades with the emergence of a smooth, large-scale, very slowly decaying, (almost) vertically sheared horizontal mode with most of its energy in the potential component — i.e. the Pearson-Linden regime.

PACS numbers: PACS number 47.52.+j

I. INTRODUCTION

Large-scale geophysical flows usually evolve under the constraints of stable stratification and rotation. Indeed, it is known that both of these constraints, individually and in concert, profoundly affect the motion of a fluid [1]. Here we restrict our attention to the stratified problem (see [2] for a review). In particular, we study the adjustment of a two-dimensional stably stratified Boussinesq fluid to imposed large-scale density perturbations.

Starting with the work of Riley, Metcalfe & Weissman [3], there have been numerous studies of three-dimensional stably stratified flows in both forced and decaying scenarios that indicate a spontaneous generation of layered structures from initially isotropic fields (see for example [4],[5],[6]). Indeed, the small Froude number limit of the three-dimensional Boussinesq equations (especially in the decaying case) indicates layerwise motion with weak vertical correlation [3],[2],[7]. Further, amongst the variety of wave-vortex and wave-wave interactions possible in the three-dimensional case [5],[8], it appears as though the near resonant transfer into the slow (horizontal) wave modes via fast waves is responsible for the generation of these layered structures [6]. In fact the partitioning of energy between the fast and slow components in the general rotating stratified system is an outstanding problem in atmospheric dynamics (for example Ford et. al [9] and the references therein). In light of this, it may be instructive to understand how energy re-distribution proceeds in the relatively simplified, two-dimensional system that no longer possesses a distinct vortex mode and presents a situation wherein one can study the interaction of wave modes in isolation. A related issue is the fact that potential vorticity is identically zero in two dimensions and a future goal is to understand if and how the constraint of potential vorticity conservation in three-dimensional flows alters wave interactions.

With regards to the two-dimensional problem, detailed numerical work was carried out by Bouruet-Aubertot, Sommeria & Staquet [10],[11] wherein a bounded domain with no-flow conditions was considered. Focussing on the instabilities suffered by prescribed flows, they showed that the wave-wave interactions result in a transfer of energy from large to small scales (see also Orlanski & Cerasoli [12]). Their analysis of imposed standing gravity waves indicated that these waves necessarily break after a finite time (depending on the initial amplitude) and the spectral redistribution of energy proceeds via the so-called wave-turbulence paradigm [13]. Here we lift the no flow condition by considering the problem in a periodic domain and pursue the time evolution, from a state of rest, of a smooth large scale density (or temperature) perturbation. As one might anticipate, the aforementioned wave-breaking is one stage in the overall scheme of things.

Apart from geophysical motivations, the system under consideration is part of an extended family of flows — the so-called dynamically *active scalars* [14],[15]. In particular, statistical properties of the neutral and unstably stratified Boussinesq systems have been the subject of recent investigations [16],[17],[18] and the present work, wherein the system additionally supports waves, can be viewed as a continuation of these efforts. Further, given the possibility of a finite-time breakdown of regularity in a two-dimensional setting, these problems are of considerable mathematical interest and the reader inclined to pursue such matters is referred to Cordoba & Fefferman [19] (and the references therein) for a recent overview.

The remainder of the paper is organized as follows : we first provide the basic equations and conservation laws along with the setup of the numerical experiment. Next we touch briefly upon the initial stages of evolution that have been well documented in existing literature. We then proceed to the focus of the paper showing a k^{-1} scaling for the potential energy spectrum followed by the establishment of a transient $k^{-5/3}$ scaling for the kinetic energy, the onset of nearly exponential decay of enstrophy and the establishment of an invariant normalized vorticity PDF. Due to the rapid decay of energy at this stage of the problem, the spectra are seen to gradually shift downwards, their scaling becomes less distinct, and finally we see the emergence of a very slowly decaying, vertically sheared, large-scale, predominantly horizontal mode — most of whose energy is trapped in the potential component — as elucidated by Pearson & Linden [20]. We conclude by summarizing the various stages, discuss the fate of stratified adjustment in the inviscid limit and point out avenues for future research.

II. THE GOVERNING EQUATIONS

The equations governing two-dimensional stratified flow under the Boussinesq approximation are [7]

$$\begin{aligned}
\frac{D\vec{u}}{Dt} &= -\frac{\nabla p'}{\rho_0} + g\alpha\theta\hat{k} \\
\frac{D\theta}{Dt} + \lambda w &= 0 ; \quad \lambda = \frac{B}{\alpha\rho_0} \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.
\end{aligned} \tag{1}$$

In the above $\rho = \rho_b + \rho'$, $p = p_b + p'$ and $T = T_b + \theta$, where $\rho_b = \rho_0 + \hat{\rho}(z)$. In particular, $\hat{\rho}(z) = -Bz$ and the basic state is in hydrostatic balance, i.e. $\partial p_b / \partial z = -\rho_b g$. The temperature and density are related via $\rho = \rho_o[1 + \alpha(T_0 - T)]$, and thus the basic state temperature profile is $T_b = T_0 + \hat{T}(z)$ with $\hat{T}(z) = -\hat{\rho}(z)/(\alpha\rho_o)$. System (1) results from the assumption $\rho', \hat{\rho}(z) \ll \rho_0$. As $\rho_o, \lambda, g, \alpha$ are constants we consider the perturbed fields $(\vec{u}, p', \rho', \theta)$ in a periodic domain [38]. In vorticity-stream form (1) reads

$$\begin{aligned}
\frac{D\omega}{Dt} &= -\alpha g \frac{\partial \theta}{\partial x} ; \quad \frac{D\theta}{Dt} = -\lambda \frac{\partial \psi}{\partial x} \\
\omega &= -\nabla^2 \psi ; \quad u = -\frac{\partial \psi}{\partial z} ; \quad w = \frac{\partial \psi}{\partial x}.
\end{aligned} \tag{2}$$

Defining $E = \int_{\mathbf{D}} [u^2 + w^2 + \frac{\alpha g}{\lambda} \theta^2]$, we see that (1) conserves E — which we refer to as the total energy of the system [39].

By linearizing (2) it can be seen that the above supports gravity waves obeying the dispersion relation

$$\sigma(\vec{k}) = \pm \sqrt{(\lambda g \alpha)} \frac{k_x}{K} ; \quad K^2 = k_x^2 + k_z^2. \tag{3}$$

Note that the presence of a uniform gravitational field in (2) breaks symmetry with respect to mirror images about the x-axis. Apart from this lack of complete parity invariance, in contrast to the usual two-dimensional scenario [22] the vorticity in (2) is not a Lagrangian invariant. This indicates that the familiar two-dimensional inverse cascade of kinetic energy is unlikely to be realised as it relies, amongst other things, on the dual conservation of energy and enstrophy [22]. Also, in comparison to the free-convective case (i.e. $\lambda = 0$) [16],[18], functions of the form $f(\theta)$ are not conserved by (2). Hence an immediate direct cascade of θ^2 is unlikely. Introducing dissipation via the usual kinematic viscosity and diffusivity (both taken to be γ) — i.e. $\gamma \nabla^2 \omega$ and $\gamma \nabla^2 \theta$ in (2) — we have

$$\begin{aligned}
\frac{\partial}{\partial t} \int_{\mathbf{D}} u^2 + w^2 &= 2\alpha g \int_{\mathbf{D}} \theta w - 2\gamma \int_{\mathbf{D}} \{|\nabla w|^2 + |\nabla u|^2\} \\
\frac{\alpha g}{\lambda} \frac{\partial}{\partial t} \int_{\mathbf{D}} \theta^2 &= -2\alpha g \int_{\mathbf{D}} \theta w - 2\gamma \frac{\alpha g}{\lambda} \int_{\mathbf{D}} |\nabla \theta|^2 \\
\frac{\partial}{\partial t} \int_{\mathbf{D}} \omega^2 &= -2\alpha g \int_{\mathbf{D}} \frac{\partial \theta}{\partial x} \omega - 2\gamma \int_{\mathbf{D}} |\nabla \omega|^2.
\end{aligned} \tag{4}$$

So for $\gamma > 0$, $\frac{\partial E}{\partial t} < 0$ and (4) imply that all fields have to die out in the limit $t \rightarrow \infty$. Of course, even though E is a monotonically decreasing function of time, the enstrophy (and higher moments of ω) can show a significant increase before finally decaying away.

III. THE NUMERICAL EXPERIMENT

To study the process of adjustment, we start with an initial state at rest and in hydrostatic balance to which is imposed a large scale θ perturbation. Various mathematical forms for the initial condition were used to check the qualitative similarity in the response of the system, in particular the results presented correspond to a Gaussian

bump ($\theta(x, z, 0) = 0.5 \exp\{-4[(x - \pi)^2 + (z - \pi)^2]\}$) in the middle of the domain. System (2) is solved at a resolution of 750×750 in a 2π periodic domain using a pseudo-spectral method de-aliased by the two-thirds rule and a fourth order Runge Kutta time stepping scheme. Defining the Froude (Fr) and Reynolds (Re) numbers as $Fr = u/NL$ and $Re = uL/\gamma$ respectively, taking N to be an $O(1)$ quantity, consider an initial θ perturbation such that the total energy is also an $O(1)$ entity. Apriori we expect the following stages of evolution : as the fluid starts from rest, initially both $Fr, Re \ll 1$ — i.e. we have a laminar flow that is strongly constrained by stratification. Since γ is small, as soon as a certain fraction of the potential energy is converted to kinetic energy implying $\max(u) \sim 1$, we have $Fr \sim 1, Re \gg 1$ — i.e. a turbulent flow that is weakly constrained by stratification. Finally, because $\gamma > 0$ the fields must eventually decay so that we enter a diffusion and stratification dominated regime. In accordance with this scenario we set the parameters in the present simulation to be $\rho_o = 1, B = 0.1, \alpha = 0.1, \gamma = 5 \times 10^{-4}$ and $N = 1$.

A. The formation of fronts and their subsequent distortion

Fig. 1 shows the evolution of energy — total, potential ($\int_{\mathbf{D}} \frac{\alpha g}{\lambda} \theta^2$) and kinetic ($\int_{\mathbf{D}} [u^2 + w^2]$) — and enstrophy with time. Focussing on the early stages of development shown in the first column of Fig. 2, as predicted from linear theory we see the generation of gravity waves that mediate the exchange of energy between the potential and kinetic components. Indeed, simulations with differing initial conditions and variations in the strength of stratification follow qualitatively similar paths though the quantitative partition of energy between the potential and kinetic components is not identical in all situations. Further, at very early times ($t < 1.5s$), all gradient fields are quite mild and the evolution is fairly inviscid. Along with this laminar evolution, Fig. 3 shows the emergence of frontal structures perpendicular to the vertical direction. Moreover, this frontal development is accompanied by increasing vertical shear and, even though this is not a steady flow, the accompanying decrease in Richardson number hints at the onset of instability [23]. In fact, as noted in previous studies such a situation necessarily leads to wave-breaking [10]. In physical space we begin to see the fronts evolving into highly convoluted sheets and the wave-wave interactions result in a redistribution energy from large to small scales [10],[12]. Of course as small-scale structures are being created, we see that the relatively inviscid behaviour seen for $t < 1.5s$ ceases and dissipation of energy begins to increase.

B. Enstrophy decay, invariant PDFs and spectral scaling

Proceeding to the focus of this communication, from Fig. 1 we notice a marked change in the behaviour of the enstrophy as the fronts become severely distorted. Specifically the enstrophy, which grew during the initial front formation and subsequent development, now decays in a fairly monotonic manner. Examining the enstrophy in detail (see the log plot in Fig. 4), immediately after the maximum we notice that the primary signature of the decay is exponential along with a secondary small amplitude modulation. In fact, Fig. 5 shows the vorticity field well after the enstrophy attains it maximum value (see the figure captions for the exact times of the snapshots); as is expected via the non-conservation of vorticity from (2), we do not see successive mergers resulting in large-scale structures, but rather the vorticity field continues to consist of distinct blobs separated by sharp ridges of concentrated enstrophy dissipation.

Motivated by a somewhat similar scenario in the decaying passive scalar problem [28],[29],[30] — where the decay (of the passive field) is purely exponential — we consider the normalized variable $X = \omega/Q^{1/2}$, where $Q = \langle \omega^2 \rangle$ and $\langle \cdot \rangle$ denotes spatial averaging. From (2) and (4), after performing a spatial average, the equation governing $\langle X^{2n} \rangle$ is

$$\frac{Q}{2n} \frac{\partial}{\partial t} \langle X^{2n} \rangle = [\gamma Q_1 + \alpha g Q_2] \langle X^{2n} \rangle - \alpha g Q^{\frac{1}{2}} \langle X^{2n-1} \frac{\partial \theta}{\partial x} \rangle - (2n-1) \gamma Q \langle X^{2n-2} (\nabla X)^2 \rangle \quad (5)$$

where $Q_1 = \langle (\nabla \omega)^2 \rangle$ and $Q_2 = \langle \omega \frac{\partial \theta}{\partial x} \rangle$. In the passive scalar problem, $X = \phi/Q_p^{1/2}$ where ϕ denotes the passive field and $Q_p = \langle \phi^2 \rangle$. In that case the purely exponential decay of $\langle \phi^2 \rangle$ (and higher moments) led to $\frac{\partial \langle X^{2n} \rangle}{\partial t} = 0$ and consequently PDF(X) attained an invariant profile [29],[30]. However in the present case, the moments in (5) inherit the secondary modulation from $\langle \omega^2 \rangle$ and fluctuate about a mean. In fact, as all the moments have the same temporal fluctuations, the entire PDF is expected to attain an invariant shape but will exhibit small shifts in

magnitude. The extracted PDFs — see Fig. 6 — are plotted in three groups. The upper panel focusses on early times ($t < 7$ s) and shows the approach to self-similarity. Interestingly, this approach is characterized by a gradual decrease in intermittency, i.e. the PDFs in the initial stages of evolution are extremely fat tailed — either stretched exponentials or power laws keeping in mind the difficulty in distinguishing between these two functions [31],[30].

The middle panel of Fig. 6 shows the PDFs for the interval $t \in [12, 28]$ s, i.e. the time during which the enstrophy experiences a modulated exponential decay. The self-similarity is evident, in fact the PDFs are now characterized by a small Gaussian core and an exponential tail. The θ field during this interval is shown in Fig. 7 — notice that even though there exists a background layering, the frontal structures that ride on this background are oriented quite randomly. In other words, irrespective of its direction a one-dimensional cut of the snapshots in Fig. 7 will encounter a frontal jump. In effect the picture that emerges is, as the frontal structures become unstable ($t \approx 1.5$ s), energy that was trapped at larger scales during the very early stages of evolution begins to fill out the entire available range of scales. This "filling out" proceeds — from $t \approx 2 - 7$ s — until one achieves a state wherein the kinetic energy spectrum is close to a $k^{-5/3}$ scaling while the potential energy continues to scale as k^{-1} (though at slightly larger scales) — this can be seen in Fig. 8 which shows the kinetic and potential energy spectra from $t \approx 7$ s onwards. Of course as is seen in Fig. 8, the spectra gradually shift downwards as the total energy is decaying, the diffusive roll-off extends to larger scales and the scaling becomes less distinct (especially in the kinetic energy) as the fields become progressively smoother. In spite of this change in the nature of the flow, the enstrophy continues to decay in a modulated exponential manner and the self-similarity in the normalized vorticity field persists throughout the interval $t \in [12, 28]$ s.

Regarding the scaling of the energy spectra, the phenomenology of stably stratified turbulence suggests the presence of two regimes [26] (see especially the discussion in Section 3 of Phillips [26]), an inertial regime wherein the kinetic energy scales isotropically as $k^{-5/3}$ [27],[26] and a buoyancy regime wherein the scaling is anisotropic and goes as k_z^{-3} [11],[26]. At the present stage, $Re \gg 1$ coupled with $Fr \sim O(1)$ implies a turbulent regime that is weakly constrained by buoyancy. This, along with the observed directional independence of the spectra, suggests the applicability of inertial range phenomenology and thus supports the $k^{-5/3}$ scaling of the kinetic energy. Given the weak buoyancy constraint, the scaling of the potential energy spectrum is dictated by the geometry of the θ -field. Specifically, as mentioned earlier even though the fronts become unstable they retain much of their identity, i.e. instead of being smoothed out, they evolve into highly distorted sheets (Fig. 7). In fact, these step-like features continue to dominate the potential energy spectrum resulting in the k^{-1} scaling [24], [25].

C. Vertically sheared horizontal flows (a Pearson-Linden regime)

Returning to the energetics of the flow, Fig. 1 indicates that at long times, i.e. $t > 40$ s, the total energy reverts to an extremely slow decay. Further, in this stage almost all of the energy in the system is in the potential component (see also the third column of Fig. 2). The minute amount of kinetic energy indicates that a dissipative linear analysis would be appropriate — in fact, we are precisely in the last stages of decaying stratified turbulence as elucidated by Pearson & Linden [20] (their analysis was more detailed with diffusivity \neq viscosity). Substituting a Fourier decomposition into the dissipative linearized form of (2), we have

$$\psi(\vec{x}, t) = \int \hat{\psi}(\vec{k}) \exp\{i(k_x x + k_z z)\} \exp\{-\gamma K^2 t\} \exp\{\pm i \frac{k_x}{K} \sqrt{(\lambda \alpha g) t}\} d\vec{k}. \quad (6)$$

The oscillatory nature of the exponential results in a dominance of the integral by modes with $k_x \approx 0$. Further, at long times a subset of these modes with the smallest rate of decay will remain. Hence we are left with a flow wherein $w \approx 0$, $u = u(z)$ and $\theta = \theta(z)$, i.e. smooth vertically sheared horizontal flows with vertical structure restricted to the smallest wavenumbers (largest scales) as is seen in Fig. 9 [40]. Regarding the vorticity field, from Fig. 4 we see that the nature of the enstrophy decay changes as we proceed into the Pearson-Linden regime — indeed, the set of PDFs in the lowermost panel of Fig. 6 correspond to these late times and as would be expected from a diffusive regime, the PDF gradually acquires more of a Gaussian form.

Note that the form of (6) does not indicate how energy is transferred into the vertically sheared horizontal modes. Indeed a study of the forced 2D case [33] suggests an inverse transfer of energy into these quasi-horizontal modes, also Bartello [32] demonstrates similar behaviour in the geostrophic energy during the process of geostrophic adjustment in

the 3D Boussinesq system. A detailed examination of this process is in progress — at this stage we can only conjecture that, much like behaviour of unstratified rotating fluids [34], at long times it is the near resonant interactions that feed energy into the slow modes ($\sigma(\vec{k}) = 0$) via interactions between fast waves ($\sigma(\vec{k}) > 0$) i.e. a slow-fast-fast transfer.

IV. SUMMARY AND CONCLUSIONS

We have studied the evolution of large-scale density (temperature) perturbations in the two-dimensional, stably stratified Boussinesq equations. The advantage of starting from a state of rest is the observation of various stages through which the system naturally evolves as governed by the Froude and Reynolds numbers. Starting from a smooth profile, we immediately observe the formation of sharp fronts resulting in a frontally dominated k^{-1} potential energy spectrum. Further, the fronts spontaneously evolve into highly convoluted sheets accompanied by a spectral re-distribution of energy that culminates in the establishment of a $k^{-5/3}$ kinetic energy spectrum. Given the rapid decay of energy at this stage, the establishment of the aforementioned scaling is followed by a gradual downward shift in the spectra, the scaling becomes less distinct as the fields become progressively smoother and finally there emerges a large scale, slowly decaying, vertically sheared, almost horizontal mode wherein most of the energy is trapped in the potential component — i.e. the Pearson-Linden regime.

With regards to the vorticity, the early stages of front formation and energy re-distribution are accompanied by a rapid increase in enstrophy. Then as the kinetic energy scaling is established we see the onset of an almost monotonic decay of enstrophy. In particular, immediately after the maximum the decay is primarily exponential with a secondary small amplitude modulation. Examining the normalized vorticity field, motivated by an analogous scenario in the decaying passive scalar problem, shows it to be characterized by an invariant exponential PDF. This almost exponential decay and the associated invariant PDF persists till we enter the Pearson-Linden regime. Finally, given the strong diffusive influence, deep into the Pearson-Linden regime the PDF relaxes towards a Gaussian profile.

An interesting aspect of this problem is the inviscid limit, i.e. $\gamma \rightarrow 0$. If the active scalar system maintains its regularity i.e. $|\nabla\theta|, |\nabla u|, |\nabla w| < \infty$, then as $\gamma \rightarrow 0$ we expect $\frac{\partial E}{\partial t} = 0$. However, this does not imply $\gamma < (\nabla\omega)^2 > 0$. In fact, given that the flow will not decay, we conjecture that the decay of enstrophy and the associated invariant PDFs will be established but the flow will never enter the Pearson-Linden regime. Further, the potential energy spectrum is expected to follow a k^{-1} scaling due to the presence of fronts, whereas the scaling of the kinetic energy is problematic due to a lack of dissipation. In fact, it is quite possible to end up with equipartition leading to a pile up of kinetic energy in the largest available wavenumbers [35]. On the other hand, if the system loses its regularity then $\frac{\partial E}{\partial t} < 0$ even in the limit $\gamma \rightarrow 0$, and should behave as in the presently studied situation with fixed $\gamma > 0$. It is worth noting that issues of a similar sort arise in the classical problem of geostrophic adjustment in the shallow water equations — as pointed out by Killworth [36] (see also Kuo & Polvani [37]) — wherein the energy deficit of the geostrophic state has implications for wave-breaking and the generation of discontinuities from initially smooth data. In the present context it would be interesting to see how the difference in energy between the initial and final states scales with diffusivity.

An obvious extension of the present work is the consideration of the fully three-dimensional (and also possibly rotating) problem — as mentioned in the introduction, the issue of balance and the spontaneous generation of imbalanced (or fast) waves is an active area of work [9]. Indeed, the qualitative similarity to certain aspects of decaying geostrophic adjustment in the rotating stratified three-dimensional Boussinesq system — elegantly elucidated by Bartello [32] — further motivates such an endeavour.

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- [38] Note that if we set $\lambda = \alpha g = 2\Omega$, (1) is equivalent to a (two component) three dimensional rotating flow that satisfies $\frac{\partial}{\partial y} = 0$ and has a rotation vector given by $(-2\Omega, 0, 0)$.

- [39] There are some subtleties associated with the definition of the energy, more precisely the pseudoenergy — see Shepherd [21] for a discussion regarding this conservation law and its related symmetry.
- [40] Previous numerical work by Bouruet-Aubertot et. al. [11] shows the potential and kinetic energy spectra to scale as k_z^{-3} , i.e. their system appears to satisfy the bouyancy range phenomenology. Referring to Fig. 3 in [11] the plotted time averaged spectra are seen to follow the k_z^{-3} scaling at fairly large times. Further, they indicate that the decay of energy is quite slow. Indeed, this is consistent with the emergence of the Pearson-Linden regime which is anisotropic, decays slowly and is characterized by smooth fields, hence much steeper spectral slopes. In our case, the Pearson-Linden regime is very smooth and does not show a power law scaling. But as our diffusivity is greater than the one used in Bouruet-Aubertot et. al. [11], it is possible that reducing γ will yield a Pearson-Linden regime that shows the bouyancy range scaling. Unfortunately, given the presence of strong fronts, to keep the spectral code well posed reducing γ involves the use of a hyper-viscosity or a high wavenumber filter. At present we have chosen not to employ such smoothening techniques and are limited to $\gamma \sim 5 \times 10^{-4}$.

List of Figures

- Figure 1 : Upper Panel : Potential, kinetic and total energy with time. Lower Panel : Enstrophy as a function of time. The simulation is carried out at 750×750 resolution with $\gamma = 5 \times 10^{-4}$. The other parameters are specified in the main text. Note that the simulation was carried out till $t \approx 175$ s, this plot only extends upto $t \approx 60$ s so as not to be dominated by the Pearson-Linden regime.
- Figure 2 : Same as Fig. (??) but the different stages are split up. Upper Panel : Potential, kinetic and total energy with time in the different stages. The first column shows the relatively inviscid evolution (till about $t \approx 2$ s) followed by the generation of small scale structures. The second column starts with well defined spectral scaling of the KE and PE and signals the onset of enstrophy decay. The third column shows the slowly decaying Pearson-Linden regime where most of the energy is in the potential component. Lower Panel : Corresponding enstrophy plots.
- Figure 3 : Snapshots of the temperature field, with reference to the Fig. (??) these are at $t = 1.39, 2.34$ seconds respectively. The emergence of the fronts perpendicular to the ambient stratification is clearly evident.
- Figure 4 : $\log(\langle \omega^2 \rangle)$ Vs. time that clearly shows a modulated exponential decay immediately after the enstrophy attains its maximum. The initial portion of the curve for very small times has been omitted for clarity. Note that at long times, as one enters the Pearson-Linden regime the rate of decay slows and its nature is quite different.
- Figure 5 : Snapshots of the vorticity field during the modulated exponential decay of enstrophy. Note the field consists of blobs of vorticity separated by ridges of intense enstrophy dissipation. We encourage the reader to view these images in color as a grey scale printout masks the sharp features.
- Figure 6 : PDFs of the normalized vorticity field. The upper panel shows the approach to a self-similar profile, note the decrease in intermittency with time. The two bunches of curves in the lower panel consists of profiles evenly spanning $t \in [12, 28]$ sec and $t \in [126, 173]$ sec respectively. The upper bunch represents the interval during which the enstrophy decay is exponential in character (Fig. ??) whereas the lower bunch shows the PDF's in the Pearson-Linden regime.
- Figure 7 : Snapshots of the θ - field corresponding to Fig. (??). Note that even though there exists a background layering, the fronts riding atop this layering are oriented in a fairly random manner. In other words, irrespective of direction a one-dimensional cut of the θ - field will encounter a frontal jump.
- Figure 8 : Spectra of the kinetic and potential energy from 1D cuts of the θ, u, w fields. The cuts are made in the vertical and horizontal directions and the scaling is identical in both the cases (these plots are the averages of the two cases). The two bunches of curves have been shifted for clarity, the upper bunch are the KE spectra with the dot-dash line showing a $k^{-5/3}$ spectrum. Similarly the lower bunch are the PE spectra with the corresponding dot-dash line showing a k^{-1} spectrum. The kinetic energy attains $k^{-5/3}$ scaling by about $t \approx 7$ s. This is clearly seen in the compensated spectrum plotted at $t = 6.7$ s, after which we see the gradual downward shift of the spectra, the extension of the diffusive roll-off to larger scales and the scaling becomes progressively less distinct. Specifically, the plots are evenly spaced in the interval $t \in [6, 10]$ s.
- Figure 9 : The establishment of a quasi-horizontal temperature field as we enter the final stages of decay — i.e. a Pearson-Linden regime. Referring to Fig. (??) we see that almost all the energy in the system is now in the potential component.